Problem 11B,9
For each $r \in [0,1)$ think of P_r as an operator on $L^2(\partial D)$.

- Show that P_r is a self-adjoint compact operator for each $r \in [0,1)$.
- For each $r \in [0,1)$, find all eigenvalues and eigenvectors of P_r.
- Prove or disprove:$\lim_{r\uparrow 1}\|I - P_r\| = 0$.

Proof.
- The fact that P_r is self-adjoint follows directly from changing the order of integration. For any sequence of functions f_n in the unit ball of $L^2(\partial D)$, we can easily find a subsequence which converges weakly. Without loss of generality, wo still denote it by $\{f_n\}_{n\in\mathbb{N}}$. Now we claim $P_r(f_n)$ is a Cauchy sequence in $L^2(\partial D)$. In fact, for any $\epsilon > 0$, there exists $N, N_1 \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} r^{2k} < \epsilon$$

and for any $m, n \geq N_1, k \in [0, N]$,

$$|\hat{f_m}(k) - \hat{f}_n(k)| < \epsilon,$$

Then it is easy to check

$$\sum_{k=-\infty}^{\infty} r^{2|k|}|\hat{f_m}(k) - \hat{f}_n(k)|^2 < C\epsilon.$$

for some universal constant C. This finishes the proof of the above claim.

- Suppose $P_r f(z) = \lambda f(z)$, then

$$\sum_{k=-\infty}^{\infty} r^{|n|}\hat{f}(n)z^n = \lambda f(z).$$

So $r^{|n|}\hat{f}(n) = \lambda\hat{f}(n)$ for all integer n. Since $r < 1$, this forces $\hat{f}(n) = 0$ which implies $f = 0$,or $\lambda = r^{|n_0|}$ for some n_0. In the latter case, the correspodning $f = cz^{n_0}$ or $f = cz^{-n_0}$. This gives all possible eigenvalues and eigenvectors.

- Regard $f = \sum_{k=-\infty}^{\infty}\hat{f}(n)z^n$, then $\|(I - P_r)f\|^2 = \sum_{k=-\infty}^{\infty}(1 - r^{|n|})^2|\hat{f}(n)|^2$. If the result is true, then the right hand side should go to zero uniformly in f. But we know that $x^n, x \in (0,1)$ is not uniformly converging to the constant 1, which implies the result is false. In fact, one can easily construct a series $r_i \to 1, f_i \in L^2(\partial D)$ to show the convergence is not uniform.
□

Problem 11B,11
Show that if $f, g \in L^1(\partial D)$ then

$$(f * g)\tilde{}(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{f}(x)\tilde{g}(t-x)dx,$$

for those $t \in \mathbb{R}$ such that $(f * g)(e^{it})$ make sense.

Proof. This follows directly from the definition. □

Problem 11B,15

Prove that if $f, g \in L^2(\partial D)$ then for every $n \in \mathbb{Z}$

$$(fg)\hat{}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\hat{g}(n-k)$$

Proof. One can easily see the result is true for z^k, which is an orthogonal basis for $L^2(\partial D)$. Then the result follows from approximating genenal function by these basis. ☐

Problem 11B,18
Prove Wirtinger's inequality: If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable 2π-periodic function and $\int_{-\pi}^{\pi} f(t)dt = 0$, then

$$\int_{-\pi}^{\pi} (f(t))^2 dt \leq \int_{-\pi}^{\pi} (f^{'}(t))^2 dt$$

with equality holds if and only if $f(t) = a\sin(t) + b\cos(t)$ for some constant a, b.

Proof. The proof uses Parseval inequality. If we use the Fourier series of the form $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$ for f, then the assumption implies $a_0 = 0$. Then the Wirtinger's inequality follows directly from Parseval inequality. The equality case also follows easily. For a slightly more general result, one can see Stein's book "Fourier Analysis" page 90-91 exercise 11. ☐

Problem 11C,5
Prove that if p is a polynomial on $\mathbb{R}$ with complex coefficients and $f : \mathbb{R} \to \mathbb{C}$ is defined by $f(x) = p(x)e^{-\pi x^2}$, then there exists a polynomial q on $\mathbb{R}$ with complex coefficients such that $deg\ q = deg\ p$ and $\hat{f}(t) = q(t)e^{-\pi t^2}$ for all $t \in \mathbb{R}$.

Proof. We only need to consider the case when $p(x) = x^k$ for some nonnegative integer k. 11.51 implies the case when $k = 0$. When $k = 1$, the calculation in 11.51 shows that

$$-2\pi i \int_{-\infty}^{\infty} xe^{-\pi x^2} e^{-2\pi ixt} dx = -2\pi t e^{-\pi t^2}.$$

This is exactlt the case $k = 1$. For $k = 2$,

$$\begin{aligned} \int_{-\infty}^{\infty} x^2 e^{-\pi x^2} e^{-2\pi ixt} dx &= \int_{-\infty}^{\infty} xe^{-2\pi ixt} d(\frac{e^{-\pi x^2}}{-2\pi}) \\ &= \int_{-\infty}^{\infty} (\frac{e^{-\pi x^2}}{2\pi})(e^{-2\pi ixt} - x(-2\pi it)e^{-2\pi ixt}) dx \end{aligned} \tag{1}$$

Now one can easily see that using the case $k = 0, 1$, we can conclude the case when $k = 2$. For more general k, the result follows from induction argument. ☐

Problem 11C,6
Suppose

$$f(x) = \begin{cases} xe^{-2\pi x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Show that $\hat{f}(t) = \frac{1}{4\pi^2(1+it)^2}$ for all $t \in \mathbb{R}$.

Proof.

$$\begin{aligned} \hat{f}(t) &= \int_0^{\infty} xe^{-2\pi x - 2\pi itx} dx \\ &= \frac{1}{4\pi^2} \int_0^{infty} xe^{-x-itx} dx \end{aligned} \tag{2}$$

Now we calculate the real part and imaginary part.

$$\begin{aligned}\int_0^\infty xe^{-x}\cos tx dx &= \int_0^\infty xe^{-x} d(\frac{\sin tx}{t}) \\ &= -\int_0^\infty (e^{-x} - xe^{-x})\frac{\sin tx}{t} dx\end{aligned} \tag{3}$$

One can easily calculate that

$$\int_0^\infty e^{-x}\sin tx dx = \frac{t}{1+t^2},$$

so we get

$$4\pi^2(Re(\hat{f}(t))) = -\frac{1}{1+t^2} - 4\pi^2(\frac{1}{t}Im(\hat{f}(t)))$$

Integration by part again just as above, then we can easily get the result. □

Problem 11C,8
Suppose $f \in L^1(\mathbb{R})$ and $n \in \mathbb{Z}^+$. Define $g : \mathbb{R} \to \mathbb{C}$ by $g(x) = x^n f(x)$. Prove that if $g \in L^1(\mathbb{R})$, then $\hat{f}$ is n times continuously differentiable on $\mathbb{R}$ and

$$(\hat{f})^{(n)}(t) = (-2\pi i)^n \hat{g}(t)$$

for all $t \in \mathbb{R}$.

Proof. We only need to note that by calculation in 11.50, we have

$$(\hat{f})'(t) = -2\pi i \int_{-\infty}^{\infty} xf(x)e^{-2\pi tx} dx.$$

Then similar argument gives the result. □

Problem 11C,9
Suppose $n \in \mathbb{Z}^+$ and $f \in L^1(\mathbb{R})$ is n times continuously differentiable and $f^{(n)} \in L^1(\mathbb{R})$. Prove that if $t \in \mathbb{R}$, then

$$(f^{(n)})\hat{}(t) = (2\pi it)^n \hat{f}(t).$$

Proof. First note by assumption, we know that for any integer $k \in [0, n]$, $\lim_{x\to infty} f^{(k)}(x) = 0$, which is easy to see by an contradiction argument. Then using the same argumnet as in 11.54, we can finish the proof. □